On the experimental foundations of the Maxwell equations

M. Haugan[1] and C. Lämmerzahl[2]
[1]Department of Physics, Purdue University, 1396 Physics Building, West Lafayette, Indiana 47907-1396, USA
[2]Faculty of Physics, University of Konstanz, Fach M674, 78457 Konstanz, Germany

February 7, 2008

Abstract

We begin by reviewing the derivation of generalized Maxwell equations from an operational definition of the electromagnetic field and the most basic notions of what constitutes a dynamical field theory. These equations encompass the familiar Maxwell equations as a special case but, in other cases, can predict birefringence, charge non-conservation, wave damping and other effects that the familiar Maxwell equations do not. It follows that observational constraints on such effects can restrict the dynamics of the electromagnetic field to be very like the familiar Maxwellian dynamics, thus, providing an empirical foundation for the Maxwell equations. We discuss some specific observational results that contribute to that foundation.

keywords: Maxwell equations, birefringence, charge conservation
PACS: 03.50 De, 04.80.-y, 41.20.-q, 03.30.+p

1 Introduction

According to special relativity and to metric theories of gravity like general relativity, the dynamics of the electromagnetic field is intimately connected with the structure of spacetime. For example, light cones can be used to identify the causal structure of spacetime in these theories, and the dynamics of the electromagnetic field combines with the quantum mechanics of charged particles to determine the behavior of atomic clocks and standards of length one uses to map out spacetime geometry. This connection between spacetime structure and the dynamics of the electromagnetic field clearly motivates the sharpest possible experimental tests of the validity of the familiar Maxwellian dynamics.

Generalized Maxwell equations provide a context in which to design and interpret such tests. Section 2 reviews the derivation of these equations from an operational definition of the electromagnetic field and the most basic notions of what constitutes a dynamical field theory. Since no geometrical spacetime structure like metric or conmnection is presumed by the derivation the generalized Maxwell equations can predict birefringence, charge non-conservation, wave damping and other effects not predicted by the familiar Maxwell equations. The familiar equations are, however, a special case of the generalized ones so that experiments which search for effects like those just mentioned can determine the extent to which that the dynamics of the electromagnetic field is or is not compatible with spacetime geometry. Section 3 discusses astronomical observations whose results constrain electromagnetic field dynamics that predict finite wave propagation speeds to be very close to the familiar Maxwellian dynamics. It also discusses effects which could be used to test

the assumption of finite wave propagation speed and the assumption that the electrodynamic field solves a well-posed Cauchy problem. Section 4 contains a few closing remarks.

2 A derivation of the Maxwell equations

The mathematical structure of the Maxwell equations have been analyzed in [1] and [2]. A way to derive the Maxwell equations by means of an operational approach [3] introduces the electromagnetic field F via consideration of charged particle interferometry. Since the phase shift for small areas is found to be proportional to an interferometer's area, $\phi \sim$ area. The proportionality factor can be defined to be the electromagnetic field F: $\phi = \frac{1}{2} F_{\mu\nu}\sigma^{\mu\nu}$ (the area is described by an antisymmetric second rank tensor). In this way we *operationally* defined the electromagnetic field.

The experimental uniqueness of the phase shift immediately requires

$$0 = dF\,. \tag{1}$$

In a $3+1$–slicing of the manifold, these equations consist in three dynamical equations $\pounds_v \bar{B} = \bar{d}\bar{E}$ (v is a "timelike" vector field) $\bar{d}\bar{B} = 0$ [3]. Another three dynamical equations are needed.

These other equations can be obtained by requiring that the Maxwell field obey an *evolutional structure* with respect to the slicing. This is equivalent to requiring an evolution equation $\pounds_v F = \lambda(F)$ that is compatible with (1). In coordinates: $\pounds_v F_{\hat{\mu}\hat{\nu}} = \lambda_{\hat{\mu}\hat{\nu}}(F)$, $\pounds_v F_{\mu 0} = \lambda_\mu(F) = \lambda^0_\mu(F) + j_\mu$, where spatial indices are denoted by a hat. The first equation is contained in (1). We call the part of $\lambda_\mu(F)$ which remains as $F \to 0$ the *source* j of electromagnetic field. The requirement that the *superposition principle* should hold leads to a linear evolution equation as well as to constraints which are fulfilled by the superposition, too.

The additional requirement that the electromagnetic field should propagate with a *finite propagation speed* implies that λ is a differential operator and that it should be of first order only [4]. That gives our generalized Maxwell equations the form

$$\partial_{[\rho}F_{\mu\nu]} = 0 \qquad \partial_\nu(\lambda_\mu{}^{\nu\rho\sigma}F_{\rho\sigma}) + \bar{\lambda}_\mu{}^{\rho\sigma}F_{\rho\sigma} = j_\mu\,. \tag{2}$$

In general, these equations show birefringence, that is, two light cones, which of course violate Local Lorentz Invariance (LLI). The requirement of a *unique light cone* implies the existence of a non–degenerate second rank tensor $g^{\mu\nu}$ so that $\lambda^{\mu\nu\rho\sigma}$ can be rewritten as [5]

$$\lambda^{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{-g}(g^{\rho[\mu}g^{\nu]\sigma} - g^{\sigma[\mu}g^{\nu]\rho}) + \theta\epsilon^{\mu\nu\rho\sigma} \tag{3}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the total antisymmetric symbol. Therefore we can rewrite the inhomogeneous Maxwell equations (2) as (D is the Riemannian covariant derivative)

$$j^\nu = D_\mu F^{\mu\nu} + \Sigma^\nu_{\mu\rho}F^{\mu\rho} \qquad \text{with} \quad \Sigma^\nu_{\mu\rho} := \bar{\lambda}^\nu_{\mu\rho} - \epsilon^{\nu\sigma}{}_{\mu\rho}\partial_\sigma\theta - \left\{ {\sigma \atop \sigma[\mu} \right\} \delta^\nu_{\rho]}\,. \tag{4}$$

The form of Σ can be restricted by requiring *charge conservation* in the form $D_\mu j^\mu = 0$. The most general Σ compatible with this requirement is

$$\Sigma_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}{}^\sigma\partial_\sigma\theta\,. \tag{5}$$

Ni [6] has shown that this coupling does not violate the weak equivalence principle for falling charges. In addition, it was established in [7] that vacuum polarization effects of the Dirac equation in a space–time with torsion leads to an effective Maxwell equation with an additional coupling of

the above form. θ can be interpreted as *axion* field which appears in the low–energy limit of string theory and is a candidate for the dark matter in the universe. These effective Maxwell equations can be obtained from a Lagrangian [1].

The coupling to θ results in a violation of Local Position Invariance. However, in a rotating and accelerating frame with rotation $\boldsymbol{\omega}$ and acceleration $\boldsymbol{a}$ (stationary situation, Newtonian potential $U(0) = 0$) the above Maxwell equations acquire the form (with the PPN parameter γ)

$$\begin{aligned} j^0 &= \boldsymbol{\nabla}\cdot\boldsymbol{E} - (\boldsymbol{a} - (1+\gamma)\boldsymbol{\nabla}U)\cdot\boldsymbol{E} - 2(\boldsymbol{\omega} - \boldsymbol{\nabla}\theta)\cdot\boldsymbol{B} \\ \boldsymbol{j} &= -\boldsymbol{\nabla}\times\boldsymbol{B} - (\boldsymbol{a} - (\gamma+1)\boldsymbol{\nabla}U)\times\boldsymbol{B} + 2(\boldsymbol{\omega} - \boldsymbol{\nabla}\theta)\times\boldsymbol{E} \end{aligned} \tag{6}$$

which means that gravity and the pseudoscalar field can be transformed away. The effects of $\boldsymbol{\nabla}\theta$ or axial torsion can be simulated by a rotation of the reference frame [8].

The coupling to $\boldsymbol{\nabla}\theta$ amounts to a non–metric Faraday–effect: If at the worldline of the source the polarization of the electromagnetic field is parallelly propagated, $D_u F = 0$, then the observer sees a rotation of the polarization along his worldline, $D_v F_{\mu\nu} = 2\,\dot{x}\,k^\rho(\partial_\tau\theta)\,\epsilon_{\rho[\mu}{}^{\sigma\tau}F_{\nu]\sigma}$. The absence of such an effect implies usual Maxwell equations coupled to a space–time metric only:

$$dF = 0 \qquad\qquad d * F = j\,. \tag{7}$$

3 Experiments testing the Maxwell equations

In principle, observation of any electromagnetic phenomenon can provide a basis for a test of the validity of the Maxwell equations, but phenomena associated with wave propagation are particularly well suited to imposing sharp constraints on the dynamics of the electromagnetic field since astronomical observations can sometimes impose limits on effects that have built up in the course of propagation over huge distances. Laboratory tests can be considered when circumstances do not allow astronomical observations to distinguish between effects caused by departures from familiar Maxwell dynamics and by other physical processes.

3.1 Tests of the generalized Maxwell equations

First we derive a wave equation from the generalized Maxwell equations (2)

$$\partial_{[\mu}j_{\nu]} = \delta^\alpha_{[\mu}\lambda_{\nu]}{}^{\kappa\rho\sigma}\partial_\alpha\partial_\kappa F_{\rho\sigma} + \widetilde{\lambda}_{[\nu\mu]}{}^{\kappa\rho\sigma}\partial_\kappa F_{\rho\sigma} + \widehat{\lambda}_{[\nu\mu]}{}^{\rho\sigma}F_{\rho\sigma}\,, \tag{8}$$

where $\widetilde{\lambda}$ and $\widehat{\lambda}$ are combinations of the original λ and $\bar{\lambda}$ and its derivatives. If $\lambda_\mu{}^{\nu\rho\sigma} = \frac{1}{2}(\delta^\rho_\mu g^{\nu\sigma} - \delta^\nu_\mu g^{\rho\sigma}) + \delta\lambda_\mu{}^{\nu\rho\sigma}$ and if the derivatives of the coefficients are assumed to be negligible, then we get from (8) for $j = 0$

$$0 = \left(\delta^\rho_\mu\delta^\sigma_\nu\Box + 2\delta\lambda_{[\mu}{}^{\kappa\rho\sigma}\partial_{\nu]}\partial_\kappa\right)F_{\rho\sigma} + 2\bar{\lambda}_{[\mu}{}^{\rho\sigma}\partial_{\nu]}F_{\rho\sigma}\,. \tag{9}$$

We assume that $\delta\lambda_\nu{}^{\nu\rho\sigma}$ as well as $\bar{\lambda}_\nu{}^{\rho\sigma}$ are small.

3.1.1 Propagation of characteristics

First we discuss the behavior of characteristics, or shock waves, predicted by (9). This is determined by this equation's principal polynomial (k is the normal of the characteristic surface)

$$0 = \det\left(\delta^\rho_{[\mu}\delta^\sigma_{\nu]}k^2 + 2\delta\lambda_{[\mu}{}^{\kappa\rho\sigma}k_{\nu]}k_\kappa\right)\,. \tag{10}$$

This prediction and the results of astrophysical observations that search for birefringence effects impose the constraint $\delta\lambda_\mu{}^{\kappa\rho\sigma} \leq 10^{-28}$ [9].

3.1.2 Propagation of plane waves

If we take the electromagnetic field to be a plane wave $F = F_0 e^{-ikx}$, and do not take the eikonal limit, then the equation (9) implies

$$0 = \det\left(\delta^\rho_{[\mu}\delta^\sigma_{\nu]}k^2 + 2\delta\lambda_{[\mu}{}^{\kappa\rho\sigma}k_{\nu]}k_\kappa + 2i\bar{\lambda}_{[\mu}{}^{\rho\sigma}k_{\nu]}\right) . \tag{11}$$

The factor i associated with the $\bar{\lambda}$ stems from a term involving a single field derivative. To first order in $\delta\lambda$ and $\bar{\lambda}$ the corresponding dispersion relation is $(\det(1+\Lambda) = 1 + \mathrm{tr}\Lambda + \mathcal{O}(\Lambda^2))$

$$0 = (k^2)^5\left(k^2 + \delta\lambda_{[\mu}{}^{\kappa\mu\nu}k_{\nu]}k_\kappa + \bar{\lambda}_{[\mu}{}^{\mu\nu}k_{\nu]}\right) + \mathcal{O}(\delta\lambda^2, \bar{\lambda}^2) . \tag{12}$$

Non–trivial solutions are given by $(k^2 = \omega^2 - \mathrm{k}^2)$

$$\begin{aligned}\omega_{1,2} &= -\frac{1}{2}((\delta\lambda_\mu{}^{0\mu\hat{\nu}} + \delta\lambda_\mu{}^{\hat{\kappa}\nu 0})k_{\hat{\nu}} + i\bar{\lambda}_\mu{}^{\mu 0}) \\ &\quad \pm\left(\mathrm{k}\left(1 - \frac{1}{2}\delta\lambda_\mu{}^{0\mu 0}\right) - \frac{1}{2}\left(\delta\lambda_\rho{}^{\hat{\mu}\rho\hat{\nu}}\frac{k_{\hat{\mu}}}{\mathrm{k}}k_{\hat{\nu}} + i\bar{\lambda}_\mu{}^{\mu\hat{\nu}}\frac{k_{\hat{\nu}}}{\mathrm{k}}\right)\right) .\end{aligned} \tag{13}$$

Note the presence of two effects: (i) an anisotropic relation between $k_{\hat{\mu}}$ and ω due to $\delta\lambda$, indicating a violation of LLI, and (ii) imaginary terms due to a trace of $\bar{\lambda}$, indicating a damping of a plane wave's intensity. This damping is independent of wave length but depends on the direction in which a wave propagates. Knowledge of the distance and intrinsic brightness of stars should permit the estimation of the strength of any such damping. In principle, tests of LLI can lead to estimates of $\delta\lambda$ and observations of damping lead to estimates of $\bar{\lambda}$ and, thus, constraints on charge non-conservation. Recall that charge conservation forced $\bar{\lambda} = 0$.

In the case of a coupling to the axion alone we find the exact dispersion relation $\omega = \pm\mathrm{k}\sqrt{1 \pm \frac{1}{\mathrm{k}}\frac{\partial\theta}{\partial t}}$ so that the observed frequencies are $\omega = k \pm \frac{\partial\theta}{\partial t} - \frac{1}{2\mathrm{k}}\left(\frac{\partial\theta}{\partial t'}\right)^2 + \mathcal{O}(\mathrm{k}^{-2})$, that is, waves with opposite helicity propagate with different speeds. The results of astronomical observations lead to the constraint $\frac{\partial\theta}{\partial t} \leq 10^{-32}$ eV [10].

Predicted effects of a hypothetical θ coupling on energy levels of atoms and existing atomic physics data imply the constraint $\nabla\theta \leq 10^{-8}\ \mathrm{m}^{-1}$ [11].

3.2 Test of well–posedness of Cauchy–problem

If for an evolution the Cauchy–problem is not well posed, then this evolution depends on its history or possesses memory. In mathematical terms this means, that, under certain circumstances, one has to pose all time derivatives up to infinite order: F, $\partial_t F$, $\partial_t^2 F$, ..., $\partial_t^n F$, ... In a first approximation one may ask, whether there is any need to pose also $\partial_t F$ in addition to F. For the usual Maxwell equations (here we assumne, for simplicity, a metric) we therefore have to add a term $\partial_t^2 F$:

$$\tilde{a}_\mu{}^{\rho\sigma}\partial_0^2 F_{\rho\sigma} + \partial_\nu F_\mu{}^\nu = j_\mu . \tag{14}$$

The addition of such a term clearly violates LLI. In addition, if such a term is present, the homogeneous Maxwell equations reduce to constraints, because they are of first order only.

This term can be analyzed by the propagation of plane waves. The dispersion relation is

$$\begin{aligned}0 &= \det\left(k^2\delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\nu]} + \tilde{a}_{[\mu}{}^{[\rho\sigma]}k_{\nu]}\omega^2\right) \\ &\approx (k^2)^5\left(\omega^2 - \mathrm{k}^2 + \frac{1}{2}\omega^2\left(\tilde{a}_0{}^{[0\hat{\nu}]}k_{\hat{\nu}]} + \tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}0]}\omega + \tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}\hat{\nu}]}k_{\hat{\nu}}\right)\right)\end{aligned} \tag{15}$$

leading to the non–trivial solutions

$$\omega_{1,2} = \pm \mathrm{k} \left(1 \mp \frac{i}{8}(\tilde{a}_0{}^{[0\hat{\nu}]} + \tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}\hat{\nu}]})k_{\hat{\nu}} - \frac{i}{8}\tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}0]}\mathrm{k}\right) \tag{16}$$

$$\omega_3 = \frac{i}{\tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}0]}} \left(2 + i\frac{1}{2}(\tilde{a}_0{}^{[0\hat{\nu}]} + \tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}\hat{\nu}]})k_{\hat{\nu}} + \frac{1}{8}(\tilde{a}_{\hat{\mu}}{}^{[\hat{\mu}0]})^2\mathrm{k}^2\right) . \tag{17}$$

The third solution has no physical relevance because it diverges for vanishing $\widetilde{a}$. From the first two solutions we conclude that a second time–derivative in the Maxwell equations will lead to an anisotropic damping of plane waves.

3.3 Test of finite propagation speed

As for the heat or the Schrödinger equation, infinite propagation speed means that the order of spatial derivatives is larger than the the order of the time derivative. In our case this means that we have as modified Maxwell equations

$$\partial_\nu F_\mu{}^\nu + \tilde{b}_\mu{}^{\hat{\kappa}\hat{\lambda}\rho\sigma}\partial_{\hat{\kappa}}\partial_{\hat{\lambda}}F_{\rho\sigma} = j_\mu \, , \tag{18}$$

which again violates LLI. Again, these equations can be analyzed through propagation phenomena. We get as dispersion relation

$$\begin{aligned} 0 &= \det\left(k^2\delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\nu]} + k_{[\mu}\tilde{b}_{\nu]}{}^{\hat{\kappa}\hat{\lambda}[\rho\sigma]}k_{\hat{\kappa}}k_{\hat{\lambda}}\right) \\ &\approx (k^2)^5\left(\omega^2 - \mathrm{k}^2 + \frac{1}{2}\left(\tilde{b}_{\hat{\mu}}{}^{\hat{\kappa}\hat{\lambda}[\hat{\mu}0]}\omega + \left(\tilde{b}_0{}^{\hat{\kappa}\hat{\lambda}[0\hat{\mu}]} + \tilde{b}_{\hat{\nu}}{}^{\hat{\kappa}\hat{\lambda}[\hat{\mu}\hat{\nu}]}\right)k_{\hat{\mu}}\right)k_{\hat{\kappa}}k_{\hat{\lambda}}\right) \end{aligned} \tag{19}$$

with the solutions

$$\omega_{1,2} = \pm\mathrm{k} - \frac{1}{4}\tilde{b}_{\hat{\mu}}{}^{\hat{\kappa}\hat{\lambda}[\hat{\mu}0]}k_{\hat{\kappa}}k_{\hat{\lambda}} \mp \frac{1}{4\mathrm{k}}\left(\tilde{b}_0{}^{\hat{\kappa}\hat{\lambda}[0\hat{\mu}]} + \tilde{b}_{\hat{\nu}}{}^{\hat{\kappa}\hat{\lambda}[\hat{\mu}\hat{\nu}]}\right)k_{\hat{\mu}}k_{\hat{\kappa}}k_{\hat{\lambda}} \tag{20}$$

showing up an anisotropic dispersive propagation thus violating LLI.

4 Conclusion

We have presented a new test theory for the dynamics of the electromagnetic field. This theory was applied to propagation phenomena well suited to testing the Maxwell equations with astronomical data. We reviewed existing constraints on conventional anomalous electromagnetic field dynamics and also discussed the effects of anomalies associated with the appearance of higher-order field derivatives in the generalized Maxwell equations. These latter anomalies cause either anisotropic wave propagation or wave damping. The structure of the test theory reviewed here is sufficiently general to provide a context for the design and interpretation of experimental tests of special relativity and of metric gravitation theories like general relativity.

C.L. thanks the Optikzentrum of the University of Konstanz for financial support.

References

[1] R.A. Puntigam, C. Lämmerzahl, and F.W. Hehl, Class. Quantum Grav. **96** (1997) 245

[2] F.W. Hehl, this volume and literature cited within.

[3] C. Lämmerzahl, R.A.S. Puntigam and F.W. Hehl, in *Proceedings of the 8th Marcel Grossmann Meeting, Jerusalem 1997*, edited by T. Piran, R. Ruffini, World Scientific, Singapore 1999, p. 457; C. Lämmerzahl, F.W. Hehl, A. Macias: *Reasons for the electromagnetic field to obey Maxwell's equations*, in preparation.

[4] V.Ya. Ivrii and V.M. Petkov, Russian Math. Surveys **29** (1974) 1

[5] W.–T. Ni: A non–metric theory of gravity, unpublished manuscript, Univ. of Bozeman 1973

[6] W.–T. Ni, Phys. Rev. Lett. **96** (1977) 245

[7] V. DeSabbata and M. Gasperini M., Phys. Rev. **D 23** (1981) 2116

[8] P. von der Heyde, Lett. Nuovo Cimento **14** (1975) 250

[9] M.P. Haugan and T.F. Kauffmann, Phys. Rev. **D 52** (1995) 3168

[10] S.M. Carrol, G.B. Field, and R. Jackiw, Phys. Rev. **D 41** (1990), 1231

[11] C. Lämmerzahl and W.–T. Ni, work in progress